

ON EQUIVARIANT ELLIPTIC GENERA OF TORIC CALABI-YAU 3-FOLDS

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Dedicated to the memory of Professor S.-S. Chern on the occasion of his 104th birthday.

ABSTRACT. We show that equivariant elliptic genera of toric Calabi-Yau 3-folds are generalized weak Jacobi forms. We also introduce a notion of averaged equivariant elliptic genera of toric Calabi-Yau 3-folds, and show that they are ordinary weak Jacobi forms given by an explicit formula predicted by Eguchi and Sugawara.

1. INTRODUCTION

Geometrically, by the elliptic genus of a compact complex manifold Y of dimension n we mean the Euler characteristic:

$$(1) \quad Z_Y(\tau, z) = \chi(Y, \text{Ell}(T^*Y; \tau, z)),$$

where for a vector bundle E on Y , $\text{Ell}(E; \tau, z) \in y^{-n/2}K(Y)[[q, y, y^{-1}]]$ is defined by:

$$(2) \quad \begin{aligned} & \text{Ell}(E; \tau, z) \\ &= q^{-n/2} \bigotimes_{m \geq 1} (\Lambda_{-yq^{m-1}}(E) \otimes \Lambda_{-y^{-1}q^m}(E^*) \otimes S_{q^m}(E) \otimes S_{q^m}(E^*)). \end{aligned}$$

Here $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$. When Y is Calabi-Yau, $Z_Y(\tau, z)$ is a weak Jacobi form of weight 0 and index $n/2$ (see e.g. [8, 2]). Since there is a unique (up to constant) weak Jacobi form of index $3/2$ given by

$$(3) \quad \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)},$$

when $n = 3$, one has

$$(4) \quad Z_Y(\tau, z) = \frac{\chi(Y)}{2} \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}.$$

Physically, elliptic genus of a compact Calabi-Yau manifold can be defined as a supersymmetric index of an $N = 2$ superconformal field theory associated to it (see e.g. [11, 4, 8]), thus it can then be expressed in terms of some $N = 2$ characters. Such expressions can often be

shown to match with the geometric definition by computing residues of some elliptic functions (see [10, 6, 7]).

For a noncompact Calabi-Yau manifold, there are still physical definition and computations of the elliptic genera. Eguchi and Sugawara [5] conjectured that (4) also holds for noncompact Calabi-Yau 3-folds. Unfortunately elliptic genera of noncompact complex manifolds have not been defined geometrically in general. To make sense of this conjecture, one has to first make a suitable definition.

In this paper we will focus on toric Calabi-Yau 3-fold and use the torus action to define equivariant elliptic genera. We will not use the full 3-torus but instead restrict to a 2-torus that preserves the holomorphic volume form. We will show that the equivariant elliptic genera of toric Calabi-Yau 3-folds are generalized weak Jacobi forms in the sense of §2.4. We will also introduce a notion of balanced toric Calabi-Yau 3-fold. A typical example is the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. We prove (4) for balanced toric Calabi-Yau 3-folds by establishing it first for the resolved conifold. For general toric CY 3-folds, we introduce a notion of averaged equivariant elliptic genera that are suitable for the purpose.

It is very interesting to compare the compact case and the noncompact case. In both cases some kind of localization formula is used. In the former it is Cauchy's residue formula, in the latter it is Atiyah-Bott's Lefschetz formula [1].

The rest of the paper is arranged as follows. In §2 we show that the equivariant elliptic genera of toric Calabi-Yau 3-folds are generalized weak Jacobi forms. In §3 we define balanced toric Calabi-Yau 3-folds and show that they satisfy (4). We also define averaged equivariant elliptic genera and show that they are given by (4). To conclude this paper, we propose a definition of elliptic genera of general noncompact complex manifolds in §4.

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2. EQUIVARIANT ELLIPTIC GENERA OF TORIC CALABI-YAU 3-FOLDS

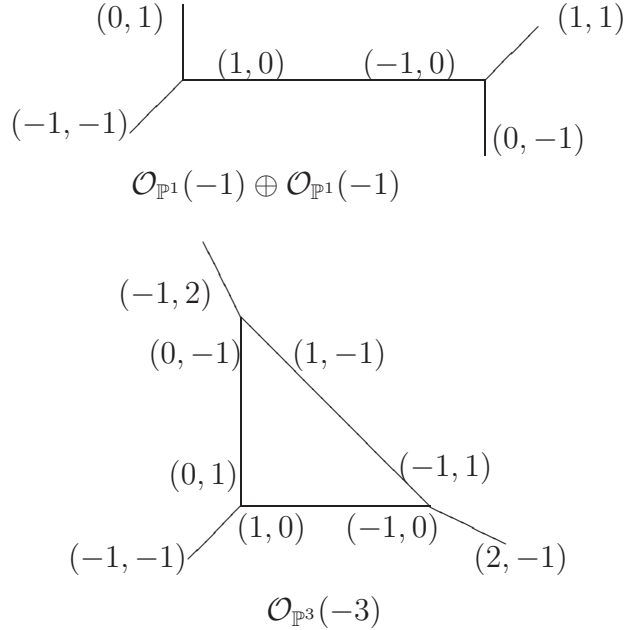
In this section we first briefly recall the definition of equivariant elliptic genera of toric Calabi-Yau 3-folds. (This is based on Atiyah-Bott-Lefschetz formula [1]. See [9] for more examples of computations

of equivariant indices using this formula.) Next we show that they are generalized weak Jacobi forms.

2.1. Toric Calabi-Yau 3-folds. Such spaces can be described by planar diagrams Γ satisfying the following conditions. Each vertex either has three incident edges (it will then be called a trivalent vertex) or has only one incident edge (it will then be called a univalent vertex). At each trivalent vertex v of Γ , the three incident outgoing edges are in the directions of three vectors $w_1^v = (a_1^v, b_1^v)$, $w_2^v = (a_2^v, b_2^v)$ and $w_3^v = (a_3^v, b_3^v)$ in \mathbb{Z}^2 respectively. These vectors will be called weight vectors at v satisfies the following conditions:

- (a) (Balancing at trivalent vertices) For all $v \in V(\Gamma)$, where $V(\Gamma)$ denotes the set of trivalent vertices of Γ , $w_1^v + w_2^v + w_3^v = 0$.
- (b) (Balancing along internal edges) For each internal edge e of Γ joining two trivalent vertices v_1 and v_2 , we have $w_{v_1,e}^{v_1} + w_{v_2,e}^{v_2} = 0$, where if e is an edge incident at a trivalent vertex v , $w_{v,e}^v$ denotes the weight vector at v along the edge e .

We understand the weight vectors as the weights of the torus action on the cotangent spaces of the fixed points. The following are two examples: They are the total spaces of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ and $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$ respectively.



2.2. Equivariant elliptic genera of toric Calabi-Yau 3-folds. We will identify the vector $w_j^v = (a_j^v, b_j^v)$ with $a_j^v t_1 + b_j^v t_2$, also denoted by w_j^v .

The equivariant elliptic genus of a toric Calabi-Yau 3-fold Y associated to a toric diagram Γ is given by

$$(5) \quad Z_Y(\tau, z; t_1, t_2) = \sum_{v \in V(\Gamma)} \prod_{j=1}^3 \frac{\theta_1(\tau, z + w_j^v)}{\theta_1(\tau, w_j^v)},$$

where θ_1 is the theta function defined by:

$$(6) \quad \begin{aligned} \theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \\ &= i q^{1/8} y^{-1/2} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1})(1 - y^{-1}q^m). \end{aligned}$$

For example,

$$(7) \quad \begin{aligned} & Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z; t_1, t_2) \\ &= \frac{\theta_1(\tau, z + t_1)}{\theta_1(\tau, t_1)} \frac{\theta_1(\tau, z + t_2)}{\theta_1(\tau, t_2)} \cdot \frac{\theta_1(\tau, z - t_1 - t_2)}{\theta_1(\tau, -t_1 - t_2)} \\ &+ \frac{\theta_1(\tau, z - t_1)}{\theta_1(\tau, -t_1)} \cdot \frac{\theta_1(\tau, z - t_2)}{\theta_1(\tau, -t_2)} \cdot \frac{\theta_1(\tau, z + t_1 + t_2)}{\theta_1(\tau, t_1 + t_2)}. \end{aligned}$$

$$(8) \quad \begin{aligned} & Z_{\mathcal{O}_{\mathbb{P}^2}(-3)}(\tau, z; t_1, t_2) \\ &= \frac{\theta_1(\tau, z + t_1)}{\theta_1(\tau, t_1)} \frac{\theta_1(\tau, z + t_2)}{\theta_1(\tau, t_2)} \cdot \frac{\theta_1(\tau, z - t_1 - t_2)}{\theta_1(\tau, -t_1 - t_2)} \\ &+ \frac{\theta_1(\tau, z - t_1)}{\theta_1(\tau, -t_1)} \cdot \frac{\theta_1(\tau, z - t_1 + t_2)}{\theta_1(\tau, -t_1 + t_2)} \cdot \frac{\theta_1(\tau, z + 2t_1 - t_2)}{\theta_1(\tau, 2t_1 - t_2)} \\ &+ \frac{\theta_1(\tau, z - t_2)}{\theta_1(\tau, -t_2)} \cdot \frac{\theta_1(\tau, z - t_2 + t_1)}{\theta_1(\tau, -t_2 + t_1)} \cdot \frac{\theta_1(\tau, z + 2t_2 - t_1)}{\theta_1(\tau, 2t_2 - t_1)}. \end{aligned}$$

2.3. Modular transformation properties of equivariant elliptic genera of toric Calabi-Yau 3-folds. Recall that the theta-function θ_1 has the following well-known modular transformation properties:

$$(9) \quad \theta_1(\tau, z + 1) = -\theta_1(\tau, z), \quad \theta_1(\tau, z + \tau) = -e^{-2\pi iz - \pi i \tau} \theta_1(\tau, z),$$

$$(10) \quad \theta_1(\tau + 1, z) = e^{\pi i/4} \theta_1(\tau, z), \quad \theta_1\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i \sqrt{\frac{\tau}{i}} e^{\frac{\pi i z^2}{\tau}} \theta_1(\tau, z).$$

From these it is straightforward to deduce the following transformation formulas:

$$\begin{aligned}
(11) \quad & Z_Y(\tau, z+1; t_1, t_2) = (-1)^3 \cdot Z_Y(\tau, z; t_1, t_2), \\
(12) \quad & Z_Y(\tau, z+\tau; t_1, t_2) = (-e^{-2\pi iz - \pi i \tau})^3 \cdot Z_Y(\tau, z; t_1, t_2), \\
(13) \quad & Z_Y(\tau, z; t_1+1, t_2) = Z_Y(\tau, z, \tau; t_1, t_2), \\
(14) \quad & Z_Y(\tau, z; t_1+\tau, t_2) = Z_Y(\tau, z; t_1, t_2), \\
(15) \quad & Z_Y(\tau, z; t_1, t_2+1) = Z_Y(\tau, z; t_1, t_2), \\
(16) \quad & Z_Y(\tau, z; t_1, t_2+\tau) = Z_Y(\tau, z; t_1, t_2), \\
(17) \quad & Z_Y(\tau+1, z; t_1, t_2) = Z_Y(\tau, z; t_1, t_2), \\
(18) \quad & Z_Y\left(-\frac{1}{\tau}, \frac{z}{\tau}; \frac{t_1}{\tau}, \frac{t_2}{\tau}\right) = e^{3 \cdot \frac{\pi i z^2}{\tau}} Z_Y(\tau, z; t_1, t_2).
\end{aligned}$$

2.4. Generalized weak Jacobi forms. We make the following definition. Suppose that $\phi : H \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic function, where H is the upper half-plane, such that

$$\begin{aligned}
(19) \quad & \phi(\tau, z+1; t_1, \dots, t_n) = (-1)^{2r} \cdot \phi(\tau, z; t_1, \dots, t_n), \\
(20) \quad & \phi(\tau, z+\tau; t_1, \dots, t_n) = (-e^{-2\pi iz - \pi i \tau})^{2r} \cdot \phi(\tau, z; t_1, \dots, t_n), \\
(21) \quad & \phi(\tau, z; t_1, \dots, t_j+1, \dots, t_n) = \phi(\tau, z; t_1, \dots, t_n), j = 1, \dots, n, \\
(22) \quad & \phi(\tau, z; t_1, \dots, t_j+\tau, \dots, t_n) = \phi(\tau, z; t_1, \dots, t_n), j = 1, \dots, n, \\
(23) \quad & \phi(\tau+1, z; t_1, \dots, t_n) = \phi(\tau, z; t_1, \dots, t_n), \\
(24) \quad & \phi\left(-\frac{1}{\tau}, \frac{z}{\tau}; \frac{t_1}{\tau}, \dots, \frac{t_n}{\tau}\right) = e^{2r \cdot \frac{\pi z^2}{\tau}} \phi(\tau, z; t_1, \dots, t_n).
\end{aligned}$$

Furthermore, ϕ is assumed to have a Fourier expansion with nonnegative powers of q . Then we say ϕ is a generalized weak Jacobi form of weight 0 and index r , with extra variables t_1, \dots, t_n . With this definition, we then have

Theorem 2.1. *The equivariant elliptic genus of a toric Calabi-Yau 3-fold is a generalized weak Jacobi form of weight 0 and index 3/2, with extra variables t_1 and t_2 .*

3. AVERAGED EQUIVARIANT ELLIPTIC GENERA OF TORIC CALABI-YAU 3-FOLDS

In this section we present some examples of toric Calabi-Yau 3-folds called balanced toric CY 3-folds whose equivariant elliptic genera are independent of t_1 and t_2 and given by the prediction of Eguchi-Sugawara. For general toric CY 3-fold we introduce a notion of averaged equivariant elliptic genera that have the same property.

3.1. The equivariant elliptic genus of the resolved conifold.
Our main result is:

Theorem 3.1. *The equivariant elliptic genus of the resolved conifold is independent of t_1 and t_2 , and it is given by:*

$$(25) \quad Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z) = \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}.$$

This is in agreement with the prediction by Eguchi-Sugawara [5, (3.48)]:

$$(26) \quad Z(\tau, z) = \frac{\chi}{2} \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)},$$

since $\chi(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) = 2$. There is a difference of a factor 1/2 with (3.39) of the same paper.

Proof. Recall

$$(27) \quad \begin{aligned} & Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z, t_1, t_2) \\ &= y^{-3/2} \cdot A(q, y, u) \cdot A(q, y, v) \cdot A(q, y, (uv)^{-1}) \\ &+ y^{-3/2} \cdot A(q, y, u^{-1}) \cdot A(q, y, v^{-1}) \cdot A(q, y, uv), \end{aligned}$$

where

$$(28) \quad A(q, y, u) = \prod_{n=1}^{\infty} \frac{(1 - yuq^{n-1})(1 - (yu)^{-1}q^n)}{(1 - uq^{n-1})(1 - u^{-1}q^n)},$$

$q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$, $u = e^{2\pi i t_1}$ and $v = e^{2\pi i t_2}$. As meromorphic functions in u , $A(q, y, u)$ and $A(q, y, u^{-1})$ have first order poles at q^n , $n \in \mathbb{Z}$, $A(q, y, u)$ and $A(q, y, u^{-1})$ have first order poles at $v^{-1}q^n$, $n \in \mathbb{Z}$. It follows that as a meromorphic function in u , $Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(q, y, u, v)$ has only simple poles at $t_1 = m\tau + n$ and $t_1 = -t_2 + m\tau + n$ ($m, n \in \mathbb{Z}$). By some straightforward calculations, one gets for all $m \in \mathbb{Z}$,

$$\begin{aligned} \text{res}_{u=q^m} A(q, z, u) du &= -q^m y^{-m} \cdot B(q, y), \\ \text{res}_{u=q^m} A(q, y, u^{-1}) du &= q^m y^m \cdot B(q, y), \\ A(q, y, uv)|_{u=q^m} &= y^{-m} A(q, y, v), \\ A(q, y, (uv)^{-1})|_{u=q^m} &= y^m A(q, y, v^{-1}), \\ A(q, y, u)|_{u=v^{-1}q^m} &= y^{-m} A(q, y, v^{-1}), \\ A(q, y, u^{-1})|_{u=v^{-1}q^m} &= y^m A(q, y, v), \\ \text{res}_{u=v^{-1}q^m} A(q, y, uv) du &= -v^{-1} q^m y^{-m} B(q, y) \\ \text{res}_{u=v^{-1}q^m} A(q, y, (uv)^{-1}) du &= v^{-1} q^m y^m B(q, y), \end{aligned}$$

where

$$(29) \quad B(q, y) = \prod_{n=1}^{\infty} \frac{(1 - yq^{n-1})(1 - y^{-1}q^n)}{(1 - q^n)^2}.$$

From these one can easily deduce that

$$(30) \quad \text{res}_{t_1=m\tau+n}(Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z, t_1, t_2)dt_1) = 0,$$

$$(31) \quad \text{res}_{t_1=-t_2+m\tau+n}(Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z, t_1, t_2)dt_2) = 0,$$

for all $m, n \in \mathbb{Z}$. It follows that $Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z, t_1, t_2)$ is holomorphic in t_1 . Since it is double periodic with periods 1 and τ , it is independent of t_1 . So is it in t_2 by the obvious symmetry between t_1 and t_2 . Therefore, $Z_{\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}}(\tau, z, t_1, t_2)$ is a weak Jacobi form of weight 0 and index $3/2$. The proof is completed by the fact that [2] the space of such forms is one-dimensional and is spanned by $\frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}$. \square

By this Theorem we obtain the following identity for theta function:

$$(32) \quad \begin{aligned} & \frac{\theta_1(\tau, z + t_1)}{\theta_1(\tau, t_1)} \frac{\theta_1(\tau, z + t_2)}{\theta_1(\tau, t_2)} \cdot \frac{\theta_1(\tau, z - t_1 - t_2)}{\theta_1(\tau, -t_1 - t_2)} \\ & + \frac{\theta_1(\tau, z - t_1)}{\theta_1(\tau, -t_1)} \cdot \frac{\theta_1(\tau, z - t_2)}{\theta_1(\tau, -t_2)} \cdot \frac{\theta_1(\tau, z + t_1 + t_2)}{\theta_1(\tau, t_1 + t_2)} \\ & = \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}. \end{aligned}$$

3.2. Balanced toric Calabi-Yau 3-folds. Suppose that Y is a toric Calabi-Yau 3-fold. If for any vertex v of its toric graph Γ , there is another vertex v' of Γ such that for suitable ordering of the weights,

$$(33) \quad w_j^v + w_j^{v'} = 0, \quad j = 1, 2, 3,$$

then we say Y is a balanced toric 3-fold. We will refer to v' as the balancing vertex of v .

There are many examples of balanced toric Calabi-Yau 3-folds. For example, the canonical line bundles of $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at two points $([1 : 0], [1 : 0])$ and $([0 : 1], [0 : 1])$, and $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at four points $([1 : 0], [1 : 0])$, $([1 : 0], [0 : 1])$, $([0 : 1], [1 : 0])$, and $([0 : 1], [0 : 1])$. Indeed, any central symmetric toric diagram corresponds to a balanced toric CY 3-fold.

3.3. Equivariant elliptic genera of balanced toric Calabi-Yau 3-folds.

Theorem 3.2. *Let Y be a balanced toric CY 3-fold. Then its equivariant elliptic genus is independent of t_1 and t_2 , and it is given by*

$$(34) \quad Z_Y(\tau, z; t_1, t_2) = \frac{\chi(Y)}{2} \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}.$$

Proof. Let Γ be the toric diagram of Y . For any vertex $v \in V(\Gamma)$, let v' be its balancing vertex. Then by (32) we have

$$\begin{aligned} Z_Y(\tau, z; t_1, t_2) &= \frac{1}{2} \sum_{v \in \Gamma} \left(\prod_{j=1}^3 \frac{\theta_1(\tau, z + w_j^v)}{\theta_1(\tau, w_j^v)} + \prod_{j=1}^3 \frac{\theta_1(\tau, z + w_j^{v'})}{\theta_1(\tau, w_j^{v'})} \right) \\ &= \frac{1}{2} \sum_{v \in \Gamma} \left(\prod_{j=1}^3 \frac{\theta_1(\tau, z + w_j^v)}{\theta_1(\tau, w_j^v)} + \prod_{j=1}^3 \frac{\theta_1(\tau, z - w_j^v)}{\theta_1(\tau, -w_j^v)} \right) \\ &= \frac{1}{2} \sum_{v \in \Gamma} \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)} \\ &= \frac{\chi(Y)}{2} \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}. \end{aligned}$$

Here in the last equality we have used the fact that $\chi(Y)$ equals the number of torus fixed points. \square

3.4. Averaged equivariant elliptic genera of toric CY 3-fold.

Inspired by the result of last section, we introduce a notion of the averaged equivariant elliptic genus of a toric CY 3-fold:

$$(35) \quad Z_Y^{av}(\tau, z; t_1, t_2) = \frac{1}{2} (Z_Y(\tau, z; t_1, t_2) + Z_Y(\tau, z; -t_1, -t_2)).$$

By almost the same proof of Theorem 3.2 one gets:

Theorem 3.3. *Let Y be a toric 3-fold. Then its averaged equivariant elliptic genus is independent of t_1 and t_2 , and it is given by*

$$(36) \quad Z_Y^{av}(\tau, z; t_1, t_2) = \frac{\chi(Y)}{2} \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)}.$$

This establishes [5, (3.48)] for general toric CY 3-folds. By comparing with Example 3 in Eguchi-Sugawara [5], our averaging of the equivariant elliptic genus seems to play the role of their assumption of the “charge conjugation symmetry”.

4. A DEFINITION OF ELLIPTIC GENERA OF NONCOMPACT COMPLEX MANIFOLDS

In this section we propose a definition of elliptic genus of a noncompact complex manifold with a compactification to a compact smooth

manifold by adding a divisor with normal crossing singularities. It is inspired by the construction of mixed Hodge structures on the cohomology nonsingular quasiprojective varieties by Deligne [3].

4.1. The case of compactification by a smooth divisor. Suppose that $D \subset M$ is a smooth divisor of a compact complex manifold M , and $U = M - D$. Denote by $\nu_{D/M}$ the normal bundle of D in M . We define

$$(37) \quad Z_U(\tau, z) = \chi(M, \text{Ell}(T^*M; \tau, z)) - \chi(D, S(\nu_{D/M}^*) \otimes \text{Ell}(T^*M|_D)).$$

The reason for making this definition is that when there is a T^2 -action on a compact 3-fold M such that D is also T^2 -invariant, then by considering the equivariant version we can get back the equivariant elliptic genus used in preceding sections.

4.2. The general case. Suppose that $D = D_1 \cup \cdots \cup D_N \subset M$ is a divisor of normal crossing singularities of a compact complex manifold M , and $U = M - D$. Denote by $\nu_{D/M}$ the normal bundle of D in M . For $I = \{i_1, \dots, i_m\} \subset \{1, \dots, N\}$, $|I| = m$, set

$$(38) \quad D_I = D_{i_1} \cap \cdots \cap D_{i_m}.$$

We define

$$(39) \quad \begin{aligned} Z_U(\tau, z) = & \chi(M, \text{Ell}(T^*M; \tau, z)) \\ & + \sum_{m=1}^N (-1)^m \sum_{|I|=m} \chi(D_I, S(\nu_{D_I/M}^*) \otimes \text{Ell}(T^*M|_{D_I})). \end{aligned}$$

We expect that with this definition, Z_U is independent of the choice of the compactification, and furthermore, when $c_1(M) = [D_1] + \cdots + [D_N]$, then Z_U is a weak Jacobi form.

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